

PBIB-DESIGNS AND ASSOCIATION SCHEMES FROM
MINIMUM SPLIT DOMINATING SETS OF CERTAIN
JUMP SIZES OF CIRCULANT GRAPHS

V. R. Kulli¹, B. Chaluvvaraju² and M. Kumara²

¹Department of Mathematics,
Gulbarga University, Gulbarga-585106, INDIA
E-mail: vrkulli@gmail.com

²Department of Mathematics,
Bangalore University,
Jnana Bharathi Campus, Bengaluru -560 056, INDIA.
E-mail: bchaluvvaraju@gmail.com, kumarabub@gmail.com

(Received: November 16, 2017)

Abstract: A dominating set D of a graph $G = (V, E)$ is a split dominating set if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set of G . The set of vertices is a γ_s - set if it is split dominating set with $\gamma_s(G)$. In this paper, we obtain the total number of γ_s -sets, the Partially Balanced Incomplete Block (PBIB)-Designs on γ_s -sets of certain jump sizes of Circulant graphs with m -association schemes for $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

Keywords and Phrases: Association schemes; PBIB Designs; Split Dominating sets; Circulant graph.

2010 Mathematics Subject Classification: 05C51, 05E30, 05C69.

1. Introduction

All graphs considered here are finite, undirected and connected with no loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G , respectively. For additional definitions and notations, the reader may refer to [7] and [11].

For a given positive integer p , let $s_1, s_2, \dots, s_t$ be a sequence of integers where $0 < s_1 < s_2 < \dots < s_t < \frac{p+1}{2}$. The Circulant graph $C_p(S)$ where $S = \{s_1, s_2, \dots, s_t\}$

is the graph on p vertices labelled as $v_1, v_2, \dots, v_p$ with vertex v_i adjacent to each vertex $v_{i \pm s_j \pmod p}$ and the values s_t are called jump sizes.

The certain jump sizes of circulant graphs are important in digital encoding; this is a wondrous technology it enables devices ranging from computers to music players to recover from errors in transmission and storage of data and restore the original data, see [15].

Bose and Nair introduced a class of binary, equi-replicate and proper designs, which are called Partially Balanced Incomplete Block (PBIB)-Designs. In these designs, all the elementary contrasts are not estimated with the same variance. The variances depend on the type of association between the objects. There are many applications of PBIB-Designs in cluster sampling, digital fingerprint codes, in architecture of web solution. For more details, we refer to [1], [5] and [6].

Given ν elements (objects or vertices), a relation satisfying the following conditions is said to be an association scheme with m classes:

- (i) Any two elements are either first associates, or second associates, $\dots$, or m^{th} associates, the relation of association being symmetric.
- (ii) Each object x has n_k k^{th} associates, the number n_k being independent of x .
- (iii) If two objects x and y are k^{th} associates, then the number of objects which are i^{th} associates of x and j^{th} associates of y is p_{ij}^k and is independent of the k^{th} associates x and y . Also $p_{ij}^k = p_{ji}^k$.

With the association scheme on ν objects, a PBIB-Design is an arrangement of ν objects into b sets (called blocks) of size g where $g < \nu$ such that

- (i) Every element is contained in exactly r blocks.
- (ii) Each block contains g distinct elements.
- (iii) Any two elements which are m^{th} associates occur together in exactly λ_m blocks.

The numbers $\nu, b, g, r, \lambda_1, \lambda_2, \dots, \lambda_m$ are called the parameters of the first kind, whereas the numbers $n_1, n_2, \dots, n_m, p_{ij}^k$ ($i, j, k = 1, 2, \dots, m$) are called the parameters of the second kind. For more details, we refer to [4].

A subset $D \subseteq V$ is said to be a dominating set of a graph G , if every vertex in $V - D$ is adjacent to some vertex in D . The minimum cardinality of vertices in

such a set is called the domination number $\gamma(G)$. For complete review, we refer to [8], [9], [10], [12] and [13].

A dominating set D of a graph G is a split dominating set if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ is the minimum cardinality of a split dominating set. The minimum split dominating set D with $|D| = \gamma_s(G)$ is called γ_s -set. This concept was introduced by Kulli and Janakiram, see [14].

Slater [16] has introduced the concept of the number of dominating sets of a graph G , which he denoted by $HED(G)$ in honor of Steve Hedetniemi. In this paper, $\#\gamma_s(G)$ is used to denote the minimum number of γ_s -sets of G . PBIB-Design associated with domination related parameters are studied by [2], [3] and [17].

2. Circulant graph $C_p(1)$

The jump size of Circulant graph is one, known as cycle C_p with $p \geq 4$ vertices. That is, $C_p(1) \cong C_p$; $p \geq 4$.

Proposition 2.1. [14] *For any Circulant graph $C_p(1)$; $p \geq 4$ vertices,*

$$\gamma_s(C_p(1)) = \left\lceil \frac{p}{3} \right\rceil.$$

Theorem 2.1. *The collection of all γ_s -sets of a Circulant graph $C_p(1)$; $p = 3n$, $n \geq 2$ vertices form a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ - association scheme and parameters are $\nu = p$, $b = 3$, $g = \left\lceil \frac{p}{3} \right\rceil$, $r = 1$ and*

$$\lambda_m = \begin{cases} 1, & \text{if } m = 3t; t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For a given Circulant graph $C_p(1)$; $p = 3n$, $n \geq 2$ vertices labeled as $v_1, v_2, \dots, v_p$. By Proposition 2.1, we have $\gamma_s(C_p(1)) = \left\lceil \frac{p}{3} \right\rceil$.

Further, the Circulant graph $C_p(1)$; $p = 3n$, $n \geq 1$ have three blocks of γ_s -sets, it implies $b = \#\gamma_s(C_p(1)) = 3$.

By Proposition 2.1, we have $g = \gamma_s(C_p(1)) = \left\lceil \frac{p}{3} \right\rceil$, where g is the number of elements contained exactly in a block.

By virtue of the above facts, we have $r = 1$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and they are k^{th} - associates

($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$), if they have k jump sizes. These associates are as shown in Table 1 along with their matrix representations.

Elements	Association scheme						
	<i>First</i>	<i>Second</i>	$\dots$	k	$\dots$	$\frac{p-1}{2}$	$\frac{p}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	$\dots$	$v_{(p-(k-1))(\bmod p)},$ $v_{(1+k)(\bmod p)}$	$\dots$	$v_{1+\frac{p-1}{2}}, v_{1+\frac{p-1}{2}+1}$	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	$\dots$	$v_{(p-(k-2))(\bmod p)},$ $v_{(2+k)(\bmod p)}$	$\dots$	$v_{2+\frac{p-1}{2}}, v_{2+\frac{p-1}{2}+1}$	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	$\dots$	$v_{(p-(k-3))(\bmod p)},$ $v_{(3+k)(\bmod p)}$	$\dots$	$v_{3+\frac{p-1}{2}}, v_{3+\frac{p-1}{2}+1}$	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(\bmod p)},$ $v_{(i+1)(\bmod p)}$	$v_{(i-2)(\bmod p)},$ $v_{(i+2)(\bmod p)}$	$\dots$	$v_{(p-(k-i))(\bmod p)},$ $v_{(i+k)(\bmod p)}$	$\dots$	$v_{(i+\frac{p-1}{2})(\bmod p)},$ $v_{(i+\frac{p-1}{2}+1)(\bmod p)}$	$v_{(i+\frac{p}{2})(\bmod p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	$\vdots$	v_{p-k}, v_k	$\vdots$	$v_{\frac{p-1}{2}}, v_{\frac{p-1}{2}+1}$	$v_{\frac{p}{2}}$

Table 1. Association schemes of $C_p(s_1, s_2, \dots, s_t)$.

By Table 1, the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p-1}{2}$ or $1 \leq i \leq \frac{p}{2} - 1$ or $n_{\frac{p}{2}} = 1$.

With the association scheme for the Table 1, we have the matrix representation of the Circulant graph $C_p(s_1, s_2, \dots, s_t)$ is

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \dots & p_{1 \frac{p-1}{2}}^k \\ p_{21}^k & p_{22}^k & \dots & p_{2 \frac{p-1}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{(\frac{p-1}{2})1}^k & p_{(\frac{p-1}{2})2}^k & \dots & p_{(\frac{p-1}{2})(\frac{p-1}{2})}^k \end{pmatrix}$$

or

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \dots & p_{1 \frac{p}{2}}^k \\ p_{21}^k & p_{22}^k & \dots & p_{2 \frac{p}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{\frac{p}{2}1}^k & p_{\frac{p}{2}2}^k & \dots & p_{\frac{p}{2} \frac{p}{2}}^k \end{pmatrix}.$$

The possible values of k in the matrix P^k are given below:

If $k = 1$, then

- (i) $p_{ij}^1 = 1$ for $1 \leq i \leq \frac{p-1}{2} - 1$ and $1 \leq i \leq \frac{p}{2} - 1, j = i + 1$.
- (ii) $p_{ij}^1 = 1$ for $1 \leq j \leq \frac{p-1}{2} - 1$ and $1 \leq j \leq \frac{p}{2} - 1, i = 1 + j$.
- (iii) $p_{ij}^1 = 1$ for $i = \frac{p-1}{2}, j = \frac{p-1}{2}$.

If $2 \leq k \leq \frac{p-3}{2}$ and $2 \leq k \leq \frac{p}{2} - 1$, then

- (i) $p_{ij}^k = 1$ for $1 \leq i \leq \frac{p-3}{2}$ and $1 \leq i \leq \frac{p}{2} - 1, i + j = k, j = k + i$ and $i + j = p - k$.
- (ii) $p_{ij}^k = 1$ for $1 \leq j \leq \frac{p-3}{2}$ and $1 \leq j \leq \frac{p}{2} - 1, i = k + j$ and $i + j = p - k$.

If $k = \frac{p-1}{2}$ and $k = \frac{p}{2}$, then

- (i) $p_{ij}^k = 1$ for $1 \leq i \leq \frac{p-3}{2}, j = \frac{p-1}{2} - i$.
- (ii) $p_{ij}^k = 1$ for $1 \leq i \leq \frac{p-1}{2}, j = \frac{p+1}{2} - i$.
- (iii) $p_{ij}^k = 2$ for $1 \leq i \leq \frac{p}{2} - 1, j = k - i$.

with other entries are all zero.

Hence the parameters of first kind are given by $\nu = p, b = 3, g = \left\lceil \frac{p}{3} \right\rceil, r = 1, \lambda_m = 1$, where m is a multiple of 3; otherwise, $\lambda_m = 0$.

Theorem 2.2. *The collection of all γ_s -sets of a Circulant graph $C_p(1); p = 3n + 1, n \geq 2$ vertices form a PBIB-Designs with $\left\lfloor \frac{p}{2} \right\rfloor$ - association scheme and parameters are $\nu = p, b = p, g = \left\lceil \frac{p}{3} \right\rceil, r = \left\lceil \frac{p}{3} \right\rceil$ and*

$$\lambda_m = \begin{cases} 2, & \text{if } m = 3t - 1; t \geq 1 \\ \left\lceil \frac{m}{3} \right\rceil - 1, & \text{if } m = 3t - 2; t \geq 1 \\ \left\lceil \frac{p}{3} \right\rceil - \left\lceil \frac{m}{2} \right\rceil, & \text{if } m = 3t; t \geq 1. \end{cases}$$

Proof. For a given Circulant graph $C_p(1); p = 3n + 1, n \geq 2$ vertices labelled as $v_1, v_2, \dots, v_p$. By Proposition 2.1, we have $\gamma_s(C_p(1)) = \left\lceil \frac{p}{3} \right\rceil$.

Further, the Circulant graph $C_p(1); p = 3n, n \geq 1$ have p blocks of γ_s -sets, it implies $b = \#\gamma_s(C_p(1)) = p$.

By Proposition 2.1, we have $g = \gamma_s(C_p(1)) = \left\lceil \frac{p}{3} \right\rceil$, where g is the number of elements contained exactly in a block.

From the above facts, we have $r = \left\lceil \frac{p}{3} \right\rceil$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and otherwise they are k^{th} -associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$), if they have k jump sizes. These associates are as shown in the above Table 1 along with their matrix representations and the possible values of k in P^k .

Hence the parameters of first kind are given by $\nu = p$, $b = p$, $g = \lceil \frac{p}{3} \rceil$, $r = \lceil \frac{p}{3} \rceil$, $\lambda_m = 2$, where $m = 3t - 1$, $t \geq 1$; $\lambda_m = \lceil \frac{m}{3} \rceil - 1$, where $m = 3t - 2$, $t \geq 1$ and $\lambda_m = \lceil \frac{p}{3} \rceil - \lceil \frac{m}{2} \rceil$, where $m = 3t$, $t \geq 1$.

Theorem 2.3. *The collection of all γ_s -sets of a Circulant graph $C_p(1)$; $p = 3n + 2$, $n \geq 2$ vertices form a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ - association scheme and parameters are $\nu = p$, $b = p$, $g = \lceil \frac{p}{3} \rceil$, $r = \lceil \frac{p}{3} \rceil$ and*

$$\lambda_m = \begin{cases} 0, & \text{if } m = 3t - 2; t \geq 1 \\ \lceil \frac{m}{3} \rceil, & \text{if } m = 3t - 1; t \geq 1 \\ \lceil \frac{p}{3} \rceil - \frac{m}{3}, & \text{if } m = 3t; t \geq 1. \end{cases}$$

Proof. For a given Circulant graph $C_p(1)$; $p = 3n + 2$, $n \geq 2$ vertices labelled as $v_1, v_2, \dots, v_p$. By Proposition 2.1, we have $\gamma_s(C_p(1)) = \lceil \frac{p}{3} \rceil$.

Further, the Circulant graph $C_p(1)$; $p = 3n + 2$, $n \geq 2$ have p blocks of γ_s -sets, it implies $b = \#\gamma_s(C_p(1)) = p$.

By Proposition 2.1, we have $g = \gamma_s(C_p(1)) = \lceil \frac{p}{3} \rceil$, where g is the number of elements contained exactly in a block.

From the above facts, we have $r = \lceil \frac{p}{3} \rceil$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are first associates, if they have jump size 1 and they are k^{th} -associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$) if they have k jump sizes. These associates are as shown in the above Table 1 along with their matrix representations and the possible values of k in P^k .

Hence the parameters of first kind are given by $\nu = p$, $b = p$, $g = \lceil \frac{p}{3} \rceil$, $r = \lceil \frac{p}{3} \rceil$, $\lambda_m = 0$, where $m = 3t - 2$, $t \geq 1$, $\lambda_m = \lceil \frac{m}{3} \rceil$, where $m = 3t - 1$, $t \geq 1$ and $\lambda_m = \lceil \frac{p}{3} \rceil - \frac{m}{3}$, where $m = 3t$, $t \geq 1$.

3. Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$

The Circulant graph with jump size $\lfloor \frac{p}{2} \rfloor$; $p \geq 4$ vertices, is $C_p(\lfloor \frac{p}{2} \rfloor)$.

Observation 3.1.

(i) $C_p(\lfloor \frac{p}{2} \rfloor) \cong C_p(1)$; $p = 2n + 1$, $n \geq 1$.

(ii) The Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$; $p = 2n$, $n \geq 1$ vertices contain n times of K_2 's and they are disconnected.

Proposition 3.1. For any Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$; $p \geq 4$ vertices,

$$\gamma_s \left(C_p \left(\left\lfloor \frac{p}{2} \right\rfloor \right) \right) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Proof. Let $C_p(\lfloor \frac{p}{2} \rfloor)$ be a Circulant graph with $p \geq 4$ vertices. Then the removal of $\frac{p}{2}$ and $\frac{(p+1)}{2}$ vertices yields a γ_s -set, say D , which is greater than or equal to $V - D$ vertices and also $\langle V - D \rangle$ is disconnected. Thus the result follows.

Theorem 3.1. The set of all γ_s -sets of a Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$; $p \geq 4$ vertices form a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ - association scheme and the parameters are $\nu = p$, $b = p$, $g = \lfloor \frac{p}{2} \rfloor$, $r = \lfloor \frac{p}{2} \rfloor$ and $\lambda_m = \lceil \frac{p}{2} \rceil - m$; $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

Proof. For a given Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$; $p = 2n$, $n \geq 2$ vertices labelled as $v_1, v_2, \dots, v_p$.

By Proposition 3.1, we have $\gamma_s(\lfloor \frac{p}{2} \rfloor) = \lfloor \frac{p}{2} \rfloor$.

Further, the Circulant graph $C_p(\lfloor \frac{p}{2} \rfloor)$; $p = 2n$, $n \geq 2$ have p blocks of γ_s -sets, it implies $b = \#\gamma_s(C_p(\lfloor \frac{p}{2} \rfloor)) = p$.

By Proposition 3.1, we have $g = \gamma_s(C_p(\lfloor \frac{p}{2} \rfloor)) = \lfloor \frac{p}{2} \rfloor$, where g is the number of elements contained exactly in a block.

Elements	Association scheme					
	First	Second	...	k	...	$\frac{p}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	...	$v_{(p-(k-1))(\text{mod } p)},$ $v_{(1+k)(\text{mod } p)}$	...	$v_{1+\frac{p}{2}}$
v_2	v_1, v_3	v_p, v_4	...	$v_{(p-(k-2))(\text{mod } p)},$ $v_{(2+k)(\text{mod } p)}$	...	$v_{2+\frac{p}{2}}$
v_3	v_2, v_4	v_1, v_5	...	$v_{(p-(k-3))(\text{mod } p)},$ $v_{(3+k)(\text{mod } p)}$	...	$v_{3+\frac{p}{2}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(\text{mod } p)},$ $v_{(i+1)(\text{mod } p)}$	$v_{(i-2)(\text{mod } p)},$ $v_{(i+2)(\text{mod } p)}$	...	$v_{(p-(k-i))(\text{mod } p)},$ $v_{(i+k)(\text{mod } p)}$	...	$v_{(i+\frac{p}{2})(\text{mod } p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	...	v_{p-k}, v_k	...	$v_{\frac{p}{2}}$

Table 2. Association schemes of $C_p(s_1, s_2, \dots, s_t)$; $p (\geq 4)$ is even.

From the above facts, we have $r = \lfloor \frac{p}{2} \rfloor$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. We consider the two distinct elements as first associates, if they have jump size k and they are $\lfloor \frac{p}{2} \rfloor^{th}$ -associates ($1 \leq k \leq \lfloor \frac{p-2}{2} \rfloor$) if they have $\lfloor \frac{p}{2} \rfloor$ jump sizes. These associates are as follows in Table 2 along with their matrix representations.

By Table 2, the parameters of second kind are given by $n_i = 2$ for $1 \leq j \leq \frac{p}{2} - 1$, $n_{\frac{p}{2}} = 1$ with the association scheme, we have the matrix representation of the Circulant graph $C_p(s_1, s_2, \dots, s_t)$ is

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1\frac{p}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2\frac{p}{2}}^k \\ \vdots & \vdots & \ddots & \vdots \\ p_{\frac{p}{2}1}^k & p_{\frac{p}{2}2}^k & \cdots & p_{\frac{p}{2}\frac{p}{2}}^k \end{pmatrix}.$$

The possible values of k in the matrix P^k are given below:

If $k = 1$, then

(i) $p_{ij}^1 = 1$ for $1 \leq i \leq \frac{p}{2} - 1, j = i + 1$.

(ii) $p_{ij}^1 = 1$ for $i = 1 + j, 1 \leq j \leq \frac{p}{2} - 1$.

If $2 \leq k \leq \frac{p}{2} - 1$, then

(i) $p_{ij}^k = 1$ for $1 \leq j \leq \frac{p}{2} - 1, i + j = k, j = k + i$ and $i + j = p - k$.

(i) $p_{ij}^k = 1$ for $1 \leq j \leq \frac{p}{2} - 1, i = k + j$ and $i + j = p - k$.

If $k = \frac{p}{2}$, then $p_{ij}^k = 2$ for $1 \leq i \leq \frac{p}{2} - 1$ and $j = k - i$ with other entries are all zero.

Hence the parameters of first kind are given by $\nu = p, b = p, g = \lfloor \frac{p}{2} \rfloor, r = \lfloor \frac{p}{2} \rfloor$ and $\lambda_m = \lfloor \frac{p}{2} \rfloor - m; 1 \leq m \leq \lfloor \frac{p}{2} \rfloor$.

4. Circulant graph with odd jump sizes

The Circulant graph with odd jump size $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor); p \geq 6$ vertices is known as a complete bipartite graph K_{p_1, p_2} where $p_1 = p_2$; that is, $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor) \cong K_{p_1, p_2}$.

Proposition 4.1. For any Circulant graph $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor); p = 4n - 2$ or $4n - 1, n \geq 2$ vertices,

$$\gamma_s \left(C_p \left(1, 3, \dots, \left\lfloor \frac{p}{2} \right\rfloor \right) \right) = \left\lceil \frac{p}{2} \right\rceil.$$

Proof. Let $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$ be a Circulant graph with $p \geq 6$ vertices labelled as $v_1, v_2, \dots, v_p$. If $D = \{v_i : i = 2k \text{ or } 2k + 1; k \geq 1\}$ is a γ_s -set with respect to $D_1 = D \cup \{v_p\}$ of $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$, then $\gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) \leq \lceil \frac{p}{2} \rceil$.

Further, if D is any γ_s -set of $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$ and hence $|D| \geq \lceil \frac{p}{2} \rceil$. Thus the result follows.

Theorem 4.1. *The collection of all γ_s -sets of a Circulant graph $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n - 2$, $n \geq 2$ vertices form a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ -association scheme and parameters are given by $\nu = p$, $b = 2$, $g = \lceil \frac{p}{2} \rceil$, $r = 1$ and*

$$\lambda_m = \begin{cases} 1, & \text{if } m = 2t; t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For a Circulant graph $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n - 2$, $n \geq 2$ vertices labelled as $v_1, v_2, \dots, v_p$.

By Proposition 4.1, we have $\gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) = \lceil \frac{p}{2} \rceil$.

Further, the Circulant graph with odd jump sizes $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n - 2$, $n \geq 2$ have two blocks of γ_s -set, it implies

$$b = \#\gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) = 2.$$

Elements	Association scheme					
	First	Second	...	k	...	$\frac{p-1}{2}$
v_1	v_p, v_2	v_{p-1}, v_3	...	$v_{(p-(k-1))(\bmod p)},$ $v_{(1+k)(\bmod p)}$	...	$v_{1+\frac{p-1}{2}}, v_{1+\frac{p-1}{2}+1}$
v_2	v_1, v_3	v_p, v_4	...	$v_{(p-(k-2))(\bmod p)},$ $v_{(2+k)(\bmod p)}$	...	$v_{2+\frac{p-1}{2}}, v_{2+\frac{p-1}{2}+1}$
v_3	v_2, v_4	v_1, v_5	...	$v_{(p-(k-3))(\bmod p)},$ $v_{(3+k)(\bmod p)}$	...	$v_{3+\frac{p-1}{2}}, v_{3+\frac{p-1}{2}+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_i	$v_{(i-1)(\bmod p)},$ $v_{(i+1)(\bmod p)}$	$v_{(i-2)(\bmod p)},$ $v_{(i+2)(\bmod p)}$	...	$v_{(p-(k-i))(\bmod p)},$ $v_{(i+k)(\bmod p)}$	...	$v_{(i+\frac{p-1}{2})(\bmod p)},$ $v_{(i+\frac{p-1}{2}+1)(\bmod p)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_p	v_{p-1}, v_1	v_{p-2}, v_2	...	v_{p-k}, v_k	...	$v_{\frac{p-1}{2}}, v_{\frac{p-1}{2}+1}$

Table 3. Association schemes of $C_p(s_1, s_2, \dots, s_t)$; p is odd.

By Proposition 4.1, we have $g = \gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) = \left\lceil \frac{p}{2} \right\rceil$, where g is the number of elements contained exactly in a block.

From the above facts, we have $r = 1$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. The two distinct elements are odd associates, if they have odd jump size and they are even associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$) if they have even jump sizes. These associates are as shown in Table 3 along with their matrix representations.

By Table 3, the parameters of second kind are given by $n_i = 2$ for $1 \leq i \leq \frac{p-1}{2}$ and $n_{\frac{p}{2}} = 1$.

With the association scheme for the Table 3, we have the matrix representation of the Circulant graph with odd jump sizes $C_p(s_1, s_2, \dots, s_t)$; $p (\geq 6)$ vertices is

$$P^k = \begin{pmatrix} p_{11}^k & p_{12}^k & \cdots & p_{1 \frac{p-1}{2}}^k \\ p_{21}^k & p_{22}^k & \cdots & p_{2 \frac{p-1}{2}}^k \\ \vdots & \vdots & \vdots & \vdots \\ p_{(\frac{p-1}{2})1}^k & p_{(\frac{p-1}{2})2}^k & \cdots & p_{(\frac{p-1}{2}) (\frac{p-1}{2})}^k \end{pmatrix}.$$

There fore the possible values of k in the matrix P^k are given below:

If $k = 1$, then

- (i) $p_{ij}^1 = 1$ for $1 \leq i \leq \frac{p-1}{2} - 1, j = i + 1$.
- (ii) $p_{ij}^1 = 1$ for $1 \leq j \leq \frac{p-1}{2} - 1, i = 1 + j$.
- (iii) $p_{ij}^1 = 1$ for $i = \frac{p-1}{2}, j = \frac{p-1}{2}$.

If $2 \leq k \leq \frac{p-3}{2}$, then

- (i) $p_{ij}^k = 1$ for $1 \leq i \leq \frac{p-3}{2}, i + j = k, j = k + i$ and $i + j = p - k$.
- (ii) $p_{ij}^k = 1$ for $1 \leq j \leq \frac{p-3}{2}, i = k + j$ and $i + j = p - k$.

If $k = \frac{p-1}{2}$, then

- (i) $p_{ij}^k = 1$, for $1 \leq i \leq \frac{p-3}{2}, j = \frac{p-1}{2} - i$.
- (i) $p_{ij}^k = 1$, for $1 \leq i \leq \frac{p-1}{2}, j = \frac{p+1}{2} - i$.

with other entries are all zero.

Hence the parameters of first kind are given by $\nu = p$, $b = 2$, $g = \left\lceil \frac{p}{2} \right\rceil$, $r = 1$, $\lambda_m = 1$; $m = 2t$ where $t \geq 1$ and otherwise $\lambda_m = 0$.

Theorem 4.2. *The collection of all γ_s -sets of a Circulant graph $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n - 1$, $n \geq 2$ vertices form a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ -association scheme and parameters are $\nu = p$, $b = p$, $g = \left\lceil \frac{p}{2} \right\rceil$, $r = \left\lceil \frac{p}{2} \right\rceil$ and $\lambda_m = \left\lceil \frac{m}{2} \right\rceil$; $m = 2t - 1$, $t \geq 1$.*

Proof. For a given Circulant graph $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n - 1$, $n \geq 2$ vertices labelled as $v_1, v_2, \dots, v_p$.

By Proposition 4.1, we have $\gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) = \left\lceil \frac{p}{2} \right\rceil$.

Further, the Circulant graph with odd jump sizes $C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p = 4n - 1$, $n \geq 2$ have p blocks of γ_s -sets, it implies

$$b = \#\gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) = p.$$

By Proposition 4.1, we have $g = \gamma_s(C_p(1, 3, \dots, \lfloor \frac{p}{2} \rfloor)) = \left\lceil \frac{p}{2} \right\rceil$, where g is the number of elements contained exactly in a block.

From the above facts, we have $r = \left\lceil \frac{p}{2} \right\rceil$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. We consider the two distinct elements odd associates, if they have odd jump size and they are even associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$) if they have even jump sizes. These associates are as shown in the above Table 3 along with their matrix representations.

Hence the parameters of first kind are given by $\nu = p$, $b = p$, $g = \left\lceil \frac{p}{2} \right\rceil$, $r = \left\lceil \frac{p}{2} \right\rceil$ and $\lambda_m = \left\lceil \frac{m}{2} \right\rceil$ for $m = 2t - 1$, $t \geq 1$.

5. Circulant graph with even jump sizes

The even jump size $(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$ of Circulant graph is denoted by $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices.

Proposition 5.1. *For any Circulant graph $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ vertices,*

$$\gamma_s \left(C_p \left(2, 4, \dots, \left\lfloor \frac{p}{2} \right\rfloor \right) \right) = 2$$

Proof. Let $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$ be a Circulant graph with $p \geq 4$ vertices. Since the Circulant graph $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$ is a $(2n - 1)$ -regular for $p \geq 4$ vertices. Hence

$\gamma_s(C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)) = 2$ follows.

Theorem 5.1. *The collection of all γ_s -sets of a Circulant graph $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$; $p \geq 4$ vertices form a PBIB-Designs with $\lfloor \frac{p}{2} \rfloor$ -association scheme and parameters are $\nu = p$, $b = p$, $g = 2$, $r = 2$ and*

$$\lambda_m = \begin{cases} 1, & \text{if } m = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. For a given Circulant graph $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$; $p \geq 4$ vertices labelled as $v_1, v_2, \dots, v_p$.

By Proposition 5.1, we have $\gamma_s(C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)) = 2$.

Further, the Circulant graph with even jump sizes $C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)$ with $p \geq 4$ have p blocks of γ_s -sets, it implies

$$b = \#\gamma_s(C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)) = p.$$

By Proposition 5.1, we have $g = \gamma_s(C_p(2, 4, \dots, \lfloor \frac{p}{2} \rfloor)) = 2$, where g is the number of elements contained exactly in a block.

From the above facts, we have $r = 2$.

To obtain the m -associates for the elements, where $1 \leq m \leq \lfloor \frac{p}{2} \rfloor$. Two distinct elements odd associates, if they have odd jump size and they are even associates ($2 \leq k \leq \lfloor \frac{p}{2} \rfloor$) if they have even jump sizes. These associates are as shown in Table 2 along with their matrix representations and the possible values of k in P^k .

Hence the parameters of first kind are given by $\nu = p$, $b = p$, $g = 2$, $r = 2$ and $\lambda_m = 1$ where $m = 1$ and $\lambda_m = 0$; otherwise.

6. Circulant graph $C_p(1, 2, \dots, \lfloor \frac{p}{2} \rfloor)$

The jump size of circulant graph is $1, 2, \dots, \lfloor \frac{p}{2} \rfloor$, known as complete graph K_p with $p \geq 3$ vertices, that is, $C_p(1, 2, \dots, \lfloor \frac{p}{2} \rfloor) \cong K_p$.

Theorem 6.1. *The collection of all γ_s -sets of a Circulant graph $C_p(1, 2, 3, \dots, \lfloor \frac{p}{2} \rfloor)$; $p \geq 3$ vertices does not form a PBIB-Designs.*

Proof. Since $C_p(1, 2, \dots, \lfloor \frac{p}{2} \rfloor) \cong K_p$; $p \geq 3$ vertices with every pair of vertices are adjacent. Hence γ_s -set of $C_p(1, 2, \dots, \lfloor \frac{p}{2} \rfloor)$ does not form.

References

- [1] Anu Sharma, Cini Varghese and Seema Jaggi, A web solution for Partially Ballanced Incomplete Block experimental designs, *Computers and Electronics in Agriculture*, 99, 132-134, 2013.
- [2] B. Chaluvaram and N. Manjunath, PBIB-Designs and association schemes arising from minimum bi-connected dominating sets of some special classes of graphs, *Afrika Matematika*, Springer, 2017.DOI: 10.1007/s13370-017-0525-5.
- [3] B. Chaluvaram, S. A. Diwakar and Shaikh Ameer Basha, PBIB-Designs with association schemes on minimum dominating sets of Circulant graphs, *Communicated*.
- [4] C. J. Colbourn, J. H. Dinitz, *Handbook of Combinatorial Designs*, CRC Press, 1996.
- [5] M. N. Das and N. C. Giri, *Designs and Analysis of Experiments*, Wiley Eastern Limited, New Delhi, 1986.
- [6] A. Dey, *Incomplete Block Designs*. World Scientific Publishing Co. Pvt. Ltd., Singapore, 2010.
- [7] F. Harary, *Graph theory*, Addison-Wesley, Reading Mass, 1969.
- [8] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, Inc., New York, 1998.
- [9] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Domination in graphs: Advanced topics*, Marcel Dekker, Inc., New York, 1998.
- [10] V. R. Kulli, *Theory of Domination in Graphs*, Vishwa International Publications, Gulbarga, India 2010.
- [11] V. R. Kulli, *College Graph Theory*, Vishwa International Publications, Gulbarga, India 2012.
- [12] V. R. Kulli, *Advances in Domination Theory I*, Vishwa International Publications, Gulbarga, India 2012.
- [13] V. R. Kulli, *Advances in Domination Theory II*, Vishwa International Publications, Gulbarga, India 2013.

- [14] V. R. Kulli and B. J. Janakiram, The split domination number of a graph, Graph theory notes of New York, XXXII: 16-19, 1997.
- [15] A. Sangeetha Devi, M. M. Shanmugapriya, Application of 2-Dominator Coloring in Graphs Using MATLAB, J. Comp. and Math. Sci. Vol.7 (4), 168-174, 2016.
- [16] P. J. Slater, The Hedetniemi number of a graph, Congr. Numer., 139: 65-75, 1999.
- [17] H. B. Walikar, H. S. Ramane, B. D. Acharya, H. S. Shekarappa and S. Arumugam, Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles, AKCE J. Graphs Combin., 4(2): 223-232, 2007.